Analysis of the Von Neumann stability criteria for a finite difference scheme for solving the 2D Riemannian wave equation

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$$
\begin{aligned}
\left[\nabla_\xi^2 - \frac{1}{\nu_\xi^2}\frac{\partial^2}{\partial t^2}\right] U_\xi &= F_\xi \\
\nabla_\xi^2 &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial \xi_i}\left(g^{ij}\sqrt{|g|}\right)\frac{\partial}{\partial \xi_j} + g^{ij}\frac{\partial^2}{\partial \xi_i \partial \xi_j} \\
\nabla_\xi^2 &= \zeta^i \frac{\partial}{\partial \xi_i} + g^{ij}\frac{\partial^2}{\partial \xi_i \partial \xi_j}
\end{aligned}
$$

Finite Difference Scheme for the Riemannanian 2D acoustic wave equation

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  Then, we have

$$\zeta^i \frac{\partial U_\xi}{\partial \xi_i} + g^{ij} \frac{\partial^2 U_\xi}{\partial \xi_i \partial \xi_j} = \frac{1}{\nu_\xi^2} \frac{\partial^2 U_\xi}{\partial t^2} + F_\xi$$

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  For a 2D scheme, we have

$$\frac{\partial^2 U_\xi}{\partial t^2} = \nu^2 \left[\zeta^1 \frac{\partial U_\xi}{\partial \xi_1} + \zeta^2 \frac{\partial U_\xi}{\partial \xi_2} + g^{11} \frac{\partial^2 U_\xi}{\partial \xi_1^2} + 2g^{12} \frac{\partial^2 U_\xi}{\partial \xi_1 \partial \xi_2} + g^{22} \frac{\partial^2 U_\xi}{\partial \xi^2} \right]$$

Finite Difference Scheme for the Riemannanian 2D acoustic wave equation

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  Take the following FD scheme

$$
\begin{aligned}
\frac{\partial^2 U}{\partial t^2} &= \frac{U^{n+1}_{v,k} - 2U^n_{v,k} + U^{n-1}_{v,k}}{(\Delta t)^2} \\
\frac{\partial U}{\partial \xi_1} &= \frac{U^n_{v+1,k} - U^n_{v-1,k}}{2\Delta \xi_1} \\
\frac{\partial U}{\partial \xi_1 \partial \xi_2} &= \frac{U^n_{v+1,k+1} - U^n_{v-1,k+1} - U^n_{v+1,k-1} + U^n_{v-1,k-1}}{2\Delta \xi_1 \Delta \xi_2} \\
\frac{\partial^2 U}{\partial \xi_1^2} &= \frac{U^n_{v+1,k} - 2U^n_{v,k} + U^n_{v-1,k}}{(\Delta \xi_1^2)} \\
\frac{\partial^2 U}{\partial \xi_2^2} &= \frac{U^n_{v,k+1} - 2U^n_{v,k} + U^n_{v,k-1}}{(\Delta \xi_2^2)}
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_1 &= v\Delta \xi_1 \\
\xi_2 &= k\Delta \xi_2 \\
t &= n\Delta t \\
U^n_{v,k} &= U(\xi_1, \xi_2, t)
\end{aligned}
$$

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  So we obtain the followin discrete equation

$$
\begin{aligned}
U^{n}_{v,k} &= 2U^{n}_{v,k} - U^{n-1}_{v,k} + (\nu\Delta t)^2[\zeta^1 \left(\frac{U^{n}_{v+1,k} - U^{n}_{v-1,k}}{2\Delta\xi_1}\right) \\
&+ \zeta^2 \left(\frac{U^{n}_{v,k+1} - U^{n}_{v,k-1}}{2\Delta\xi_2}\right) \\
&+ g^{11} \left(\frac{U^{n}_{v+1,k} - 2U^{n}_{v,k} + U^{n}_{v-1,k}}{(\Delta\xi_1)^2}\right) \\
&+ g^{22} \left(\frac{U^{n}_{v,k+1} - 2U^{n}_{v,k} + U^{n}_{v,k-1}}{(\Delta\xi_2)^2}\right) \\
&+ g^{12} \left(\frac{U^{n}_{v+1,k+1} - U^{n}_{v-1,k+1} + U^{n}_{v+1,k-1} + U^{n}_{v-1,k-1}}{2\Delta\xi_1\Delta\xi_2}\right)]
\end{aligned}
$$

Von Neumann stability criteria

Von Neumann stability criteria

Consider the exact solution to the Riemannian acoustic wave equation as

$$U(\xi_1, \xi_2, t) = U^n_{v,k},$$

and the approximate solution as

$$u(\xi_1, \xi_2, t) = u^n_{v,k}.$$

The Von Neumann criteria states that

$$\varepsilon^n_{v,k} = U^n_{v,k} - u^n_{v,k},$$

where $\varepsilon^n_{v,k}$ is the error of the wavefield at (ξ_1, ξ_2, t). Suppose that we can decompose the error in terms of Fourier modes as

$$\varepsilon^n_{v,k} = e^{-i(\omega n \Delta t - k_{\xi_1} v \Delta \xi_1 - k_{\xi_2} k \Delta \xi_2)}$$

Von Neumann stability criteria

Inserting this aproximation into the discretized wave equation, we get

$$
\begin{aligned}
e^{-i\omega\Delta t}u^n_{v,k} &= 2u^n_{v,k} - e^{-i\omega\Delta t}u^n_{v,k} \\
&+ u^n_{v,k}(\nu\Delta t)^2 \frac{\zeta^1}{2\Delta\xi_1}\left(e^{ik_{\xi_1}\Delta\xi_1} - e^{-ik_{\xi_1}\Delta\xi_1}\right) \\
&+ \frac{\zeta^2}{2\Delta\xi_2}\left(e^{ik_{\xi_2}\Delta\xi_2} - e^{-ik_{\xi_2}\Delta\xi_2}\right) \\
&+ \frac{g^{11}}{(\Delta\xi_1)^2}\left(e^{ik_{\xi_1}\Delta\xi_1} - e^{-ik_{\xi_1}\Delta\xi_1} - 2\right) \\
&+ \frac{g^{22}}{(\Delta\xi_2)^2}\left(e^{ik_{\xi_2}\Delta\xi_2} - e^{-ik_{\xi_2}\Delta\xi_2} - 2\right) \\
&+ \frac{g^{12}}{2\Delta\xi_1\Delta\xi_2}(e^{ik_{\xi_1}\Delta\xi_1}e^{ik_{\xi_2}\Delta\xi_2} \\
&- e^{-ik_{\xi_1}\Delta\xi_1}e^{ik_{\xi_2}\Delta\xi_2} \\
&- e^{ik_{\xi_1}\Delta\xi_1}e^{-ik_{\xi_2}\Delta\xi_2} + e^{-ik_{\xi_1}\Delta\xi_1}e^{-ik_{\xi_1}\Delta\xi_1})
\end{aligned}
$$

Von Neumann stability criteria

After some simplifications, we get

$$\Delta t \leq \frac{1}{\nu}\left(\frac{G-h}{(\Delta\xi)^2} - \frac{Z}{4\Delta\xi}\right)^{\frac{-1}{2}},$$

where

$$\begin{aligned} G &= |g^{11} + g^{22} + 2g^{12}| \\ h &= 2|g^{12}| \\ Z &= |\zeta^1 + \zeta^2| \end{aligned}$$

Note that in the cartesian case $g_{ij} = \delta_{ij}$ we have

$$\begin{aligned} G &= 2 \\ h &= 0 \\ Z &= 0 \end{aligned}$$

and then

$$\Delta t \leq \frac{\Delta\xi}{\nu\sqrt{2}},$$

which is the standar Courant stability criteria for FD 2D wave equation in Cartesian coordinates.

Some references

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